



On Whether Angular Momentum in Electric and Magnetic Fields Radiates to Infinity

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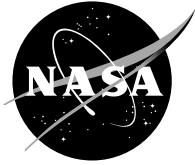
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ABSTRACT

The Feynman Disk experiment and a related thought experiment with a static magnetic field and capacitor are studied. The mechanical torque integrated over time (angular impulse) is related to the angular momentum in the electric/magnetic field. This is not called an electromagnetic field since quasi-static as well as electromagnetic effects are included. The angular momentum in the electric/magnetic field is examined to determine its static and radiative components. This comparison was then examined to see if it clarified the Abraham-Minkowski paradox.

BACKGROUND

The "Feynman Disk" experiment is described in the Feynman Lectures on Physics¹ in Chapter 17. This "Thought Experiment" shows that mechanical angular momentum by itself is not conserved. It illustrates that the conserved quantity is the sum of the mechanical angular momentum and the angular momentum in the electric/magnetic fields. This thought experiment is supplemented by a closely related thought experiment which uses a static magnetic field, and several related problems are analyzed. The controversy over the Abraham and Minkowski² formulations is well over fifty years old, and physical experiments have been performed over the years in an attempted to resolve it. This controversy involves whether adding materials with properties such as dielectric and the magnetic permeabilities changes the angular impulse in such experiments. Realizations of these thought experiments attempt to measure extremely small forces. Although some measurements have been made which seem to be above the noise level, more skeptical observers can reasonably still claim that the results are inconclusive.

One reason these issues have not been resolved theoretically is the possibility of hidden momentum. That is, even static fields in free space can carry momentum. Often this momentum is unexpected and labeled hidden momentum. In dielectric materials and magnetic materials it can be especially difficult to determine whether, and how much hidden momentum is present. Many papers have been written on this subject, for example see [3,4] and references therein.

Feynman's thought experiment¹ uses a magnetic field due to an electromagnet and a static field due to charges placed on a disk. The magnetic field is produced by an electric current going around a coil or equivalently around a single loop of wire. If the

current is changed, this creates a tangential electric field. That field acts on the charges and causes the disk to rotate (Figure 1).

The exact form of the magnetic field of a single circular current loop is somewhat complicated. If the loop is made infinitely small, the resulting field is that of a dipole⁵ and the calculation simplifies. However, to find the direction of the fields, this simplification is not needed; that is easy to find for a circular current loop. The vector potential has only a $\hat{\phi}$ component (this is shown in the section, “Torque Analysis for Both Experiments,” below). Assuming that the current loop has no net charge, the electric field must then also be in the $\hat{\phi}$ direction and proportional to the time derivative of the vector potential.

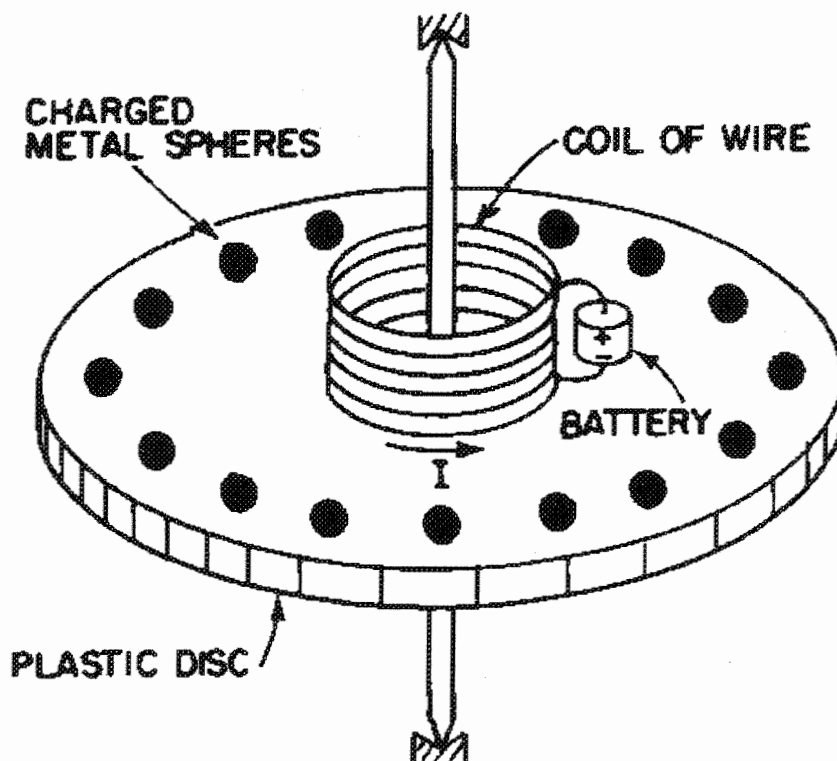


Fig. 17-5. Will the disc rotate if the current I is stopped?

Figure 1 The Feynman Disk Experiment from Chapter 17 of the Feynman Lectures.

This experiment uses a static electric field due to charges and a varying magnetic field due to an electric current. This may be abstracted to a thought experiment involving two rings. The inner ring carries a current but no net charge. The current will be chosen so that at any time, the magnitude of the current is the same everywhere on the ring.

Initially, it will also be assumed that the current varies sinusoidally. The outer ring carries a net charge but no current. This may be shown schematically as in Figure 2 below.

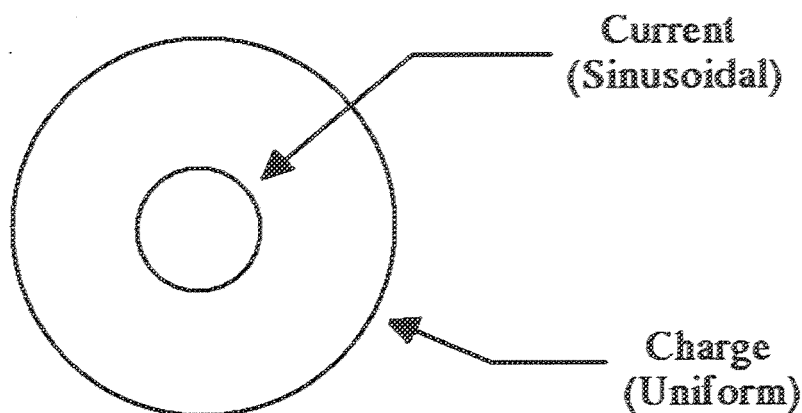


Figure 2. Schematic form of an abstraction of the Feynman Disk Experiment.

The calculation of the torque on the outer ring due to the changing current on the inner ring is straightforward, and is simpler than the calculation of angular momentum produced in the electric/magnetic fields. When the magnetic field is changed, the electric field that is produced at the outer ring is parallel to the outer ring. That produces a torque that will cause the disk to rotate if it is on low-friction bearings.

One can design a variant of this experiment by using a permanent magnet and moving charges. To do so, one might use a permanent magnet and a capacitor. This uses the Lorentz force to produce a torque. Figure 3 shows such an experimental set up. Here, the disk holds a “horseshoe” magnet; a cylindrical capacitor is placed in the field of that magnet. The magnet is arranged vertically so that each end is by an end of the capacitor and the axis of that capacitor runs along the axis of the disk

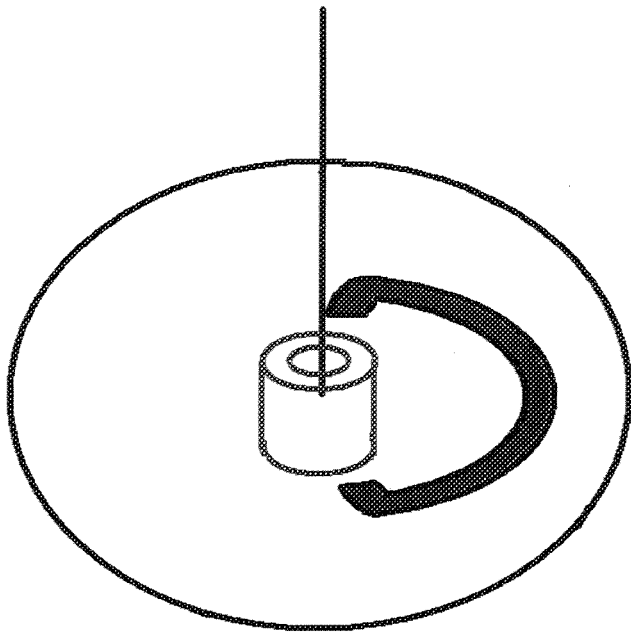


Figure 3. An alternate experiment: a disk with a permanent magnet and a cylindrical capacitor.

A slightly better arrangement would use a rotationally symmetric magnet. This can be produced by rotating the horseshoe magnet about the vertical axis (the same axis as for the cylindrical capacitor). This encloses the capacitor. Such a magnet has a much stronger magnetic field in its interior than its exterior. This decreases the unwanted coupling of the experiment from its surroundings. When a circuit for charging and discharging the capacitor is included, the result is as shown in Figure 4.

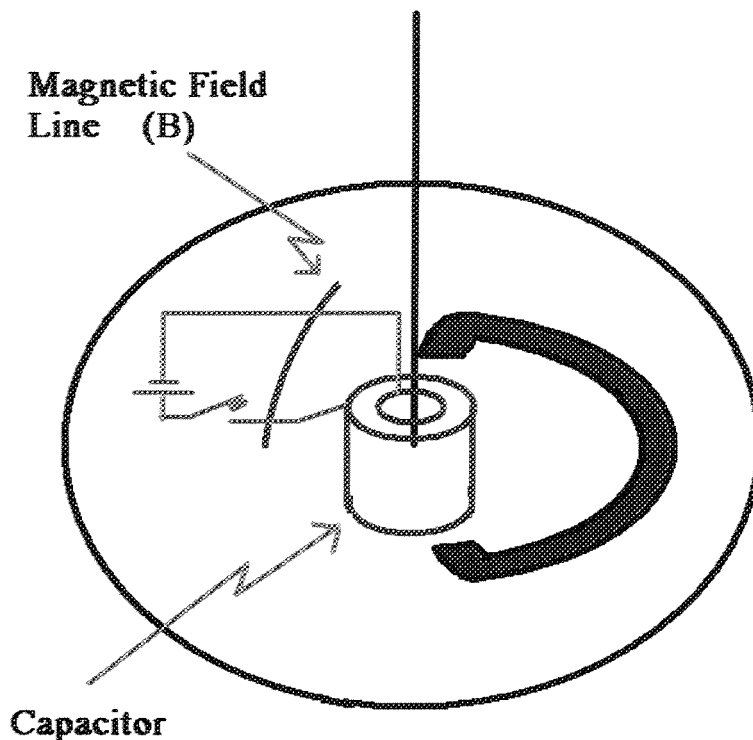


Figure 4. Variant of the Feynman Disk Experiment: a permanent magnet and a capacitor mounted on a disk with a charging/discharging circuit.

This experimental design attempts to remove sources of external interference. It is possible to design such a magnet so that the magnetic field in its exterior is orders of magnitude smaller than in its interior. The forces that one reasonably expects to produce are very small. For reasonable magnet strengths and capacitor sizes and for a disk with a diameter on the order of ten centimeter, one discharging of the capacitor results in a rotational kinetic energy comparable to $(1/2) kT$ at room temperature. That is, this energy is comparable to the thermal energy in one degree of freedom.

This design seems more practical for actual measurements than the original Feynman Disk (thought) experiment for several reasons. The permanent magnet on board (as compared to an electromagnet with current source) reduces energy requirements. It also produces a stronger magnetic field. Coupling with the exterior is also minimized, to reduce noise sources. This is an extremely important experimental issue, since such small forces are being measured.

SCALING ANALYSIS FOR RADIATED ANGULAR MOMENTUM

It is generally accepted that the total of mechanical and electromagnetic momentum is conserved. Thus, these two experiments may be analyzed by determining

the angular momentum they produce in the electric/magnetic field. We are careful not to call this an electromagnetic field, since that term would imply one field that propagates in a self sustaining way due to a changing electric field producing a magnetic field and vice versa. The angular momentum here is due to a combination of electric and the magnetic fields. However, the relevant parts of these fields are often due to independent sources, as is the case for the experiments studied here. Thus, we say that the angular momentum is produced by the electric/magnetic fields.

The angular momentum that is produced may occur in two parts. One part is contained locally while another part may radiate to infinity. If one is interested in producing a useable force (torque) and repeatedly producing that force (torque) in the same direction, then radiating momentum to infinity becomes important. Thus, the issue of radiated momentum (angular momentum) to infinity has an interest that goes beyond the analysis of these two experiments and beyond the Abraham/Minkowski paradox.

There is reason to be concerned that the variant experiment, with its static and largely contained magnetic field, cannot radiate angular momentum to infinity. This issue is simple to examine. The momentum density in the electric/magnetic field is the Poynting vector divided by the speed of light squared, so the related force, the derivative of the momentum with respect to time, is:

$$\vec{F} = \frac{1}{4\pi c} d/dt \int \vec{E} \times \vec{B} dV = \frac{1}{4\pi c} \int [d\vec{E}/dt \times \vec{B} + \vec{E} \times d\vec{B}/dt] dV$$

This shows that a force can be produced by varying either \vec{E} or \vec{B} . Using the formula for the angular momentum produced about the disk axis, one immediately sees that for both proposed experiments the fields are oriented properly to produce an angular momentum about the axis of the disk. Thus, we may estimate the variation with radial distance of the time derivative of the radiated angular momentum within a spherical shell of thickness

Δr . If this angular momentum (which is in the \hat{z} direction) is labeled as $AM(r)$, the result is then:

$$AM(r) = \frac{1}{4\pi c} \Delta r \oint r (\vec{E} \times \vec{B})_{\phi} r^2 \sin^2(\theta) d\phi d\theta$$

For the experiment proposed by Feynman (Fig. 1) the static electric field \vec{E} scales as r^{-2} . This is the behavior of a monopole. The radiating field due to the changing current on the current loop creates a magnetic field that scales as r^{-1} . The result is that for the Feynman Disk experiment the angular momentum in a shell at radius r , $AM_F(r)$, scales as

$$AM_F(r) = \frac{1}{4\pi c} \Delta r \oint r^{-2} r^{-1} r^3 f(\theta, \phi)^2 \sin^2(\theta) d\phi d\theta$$

The time derivative of the angular momentum also scales this way as a function of r . Notice that $AM_F(r)$ is asymptotically independent of r .

Next, consider the variant experiment, as shown in Figure 4. There, the radiating electric field is due to the capacitor charging and discharging. That electric field \vec{E} (and also its time derivative) scales as $(1/r)$. The static magnetic field due to the permanent magnet scales as $(1/r^3)$, since for large distances its behavior is that of a magnetic dipole. The result is that for the variant experiment the angular momentum in a shell at radius r , $AM_V(r)$, scales as

$$AM_V(r) = \frac{1}{4\pi c} \Delta r \oint r^{-1} r^{-3} r^3 f(\theta, \phi) \sin^2(\theta) d\phi d\theta = C'/r$$

The results for the two experiments differ. The reason is that each experiment involves a quantity that decays as $1/r$ (because it involves radiation) but the other quantity is monopole like for the Feynman Disk experiment and is dipole like for the variant experiment.

This conclusion about the decay rates for large r applies to two quantities. The first is the angular momentum in a thin shell of thickness dr at a (large) radius of r . The second quantity is the time derivative of the angular momentum passing through a spherical surface at radius r . Both quantities must decay at least as fast as $1/r$ for the variant experiment. For the original Feynman Disk experiment, both quantities appear to be constant independent of r . However, to be careful the possibility must be considered that the result could be smaller due to some cancellation in the integral. Thus, for the Feynman Disk experiment this will be considered more carefully in the later section, "Computation of Radiated Angular Momentum." In that section, the leading terms are found for the radiated angular momentum at a large distance using a sinusoidal current on the inner ring.

TORQUE ANALYSIS FOR BOTH EXPERIMENTS

For the original experiment as proposed by Feynman in his lectures, the calculation of a torque is straightforward. The vector potential for the current (see Ref. [6], Eqn. 5.36) is

$$\vec{A}_\phi(r, \theta, \phi) = \frac{Ia}{c} \int_0^{2\pi} \frac{\cos\phi' d\phi'}{\sqrt{a^2 + r^2 - 2ar \sin\theta \cos\phi'}}$$

This result is independent of ϕ in spherical coordinates. On the disk, $\theta = \pi/2$ so $\sin\theta = 1$.

The radius of the ring is "a" and "c" is the speed of light. We assume that the current \vec{I} is varied according to

$$\vec{I}(t, \phi) = \hat{\phi} I_0 \sin(\omega t)$$

Here \vec{I} is used for the current rather than \vec{J} to indicate that the integral over the two transverse directions has already been performed. In computing radiated quantities at the position \mathbf{r} , the “retarded” time must be used so the current that is used is

$$\vec{I}(t, \phi) = \hat{\phi} I_0 \sin(\omega(t - |\mathbf{a} - \mathbf{r}|/c))$$

Here the vector \mathbf{a} is the position on the ring at the angle ϕ . Since there is no net charge anywhere on the ring, the electric field that is produced reduces to

$$E_{\phi}(r, \theta, \phi) = -\frac{1}{c} \partial A_{\phi}(r, \theta) / \partial t = -\partial I / \partial t \frac{a}{c^2} \int_0^{2\pi} \frac{\cos \phi' d\phi'}{\sqrt{a^2 + r^2 - 2ar \sin \theta \cos \phi'}}$$

This elliptic integral can be easily computed (numerically) to give the electric field at the outer (charged) ring. The electric field is non zero and it oscillates sinusoidally in time. Multiplying this electric field by the radius r and by the total charge on the outer ring gives the resulting torque. That torque is therefore also non zero and it oscillates sinusoidally in time.

Now let us turn our attention to whether the torque is also non zero for the variant experiment shown in Figure 4. We will assume that the magnet is rotationally symmetric. Across the capacitor there is a displacement current. Including the displacement current, the current traverses a closed path following the circuit shown. Each current element produces a Lorentz force, which results in a torque about the axis of the disk. One could ask if the fact that the current follows a closed path constrains the resulting total torque. The wire may be arranged many ways. The circuit may even pass through the magnet, provided that the current follows a closed path. Is it possible that even with this freedom in the choice of a path the total torque is zero for all possible closed circuits?.

We begin by examining the properties of a magnet which is equivalent to that produced by a circular current loop lying in the plane (i.e. where $\theta = \pi/2$). The vector potential at some observation point may be computed by integrating the current vector (for the current in this current loop). One must integrate the current vector divided by the distance to the observation point. The formula for this was given above in the context of the original experiment. If the observation point is at $\phi = 0$, then the sources at $+\phi'$ combine to cancel the other components of the vector potential. This gives a vector potential lying only in the $\hat{\phi}$ direction. This is true even for observation points where $z \neq 0$, i.e. out of the plane of the current loop. The magnetic field of any rotationally symmetric permanent magnet can be produced by a collection of these current loops. Therefore, it must be possible to write the vector potential for such a permanent magnet as one having only a $\hat{\phi}$ component. That fact will be used below.

If the wires in Figure 4 lie in a plane with ϕ' constant, then the torque produced is

$$\vec{\tau} = \frac{I}{c} \oint \rho (-\vec{\nabla} \times \vec{A}) \times d\vec{l}$$

$$\vec{\tau} = \frac{I}{c} \oint \rho \{ \hat{\rho} d(A_\phi)/dz - \hat{z}(1/\rho) d(\rho A_\phi)/d\rho \} \times d\vec{l}$$

$$\vec{\tau} = \frac{I}{c} \oint \{ \hat{\rho} d(\rho A_\phi)/dz - \hat{z} d(\rho A_\phi)/d\rho \} \times \{ \hat{\rho} dl_\rho + \hat{z} dl_z \}$$

$$\vec{\tau} = \frac{I}{c} \oint \hat{\phi} \vec{\nabla}(-\rho A_\phi) \bullet d\vec{l} = 0$$

The last line follows from the fact that the integral of the gradient of any function around a closed path is zero. This shows that a torque cannot be produced when the wires lie in a plane with ϕ constant. It is straightforward to show that this result also holds when the wires are not in such a plane. Basically the proof uses the fact that a current in the $\hat{\phi}$ direction cannot produce a torque, and by showing that the $\hat{\rho}$ and \hat{z} components of a path produce a torque which is independent of whether there is also a $\hat{\phi}$ component.

To summarize the results of this section it was found that the original experiment does produce a torque although the variant experiment cannot produce a torque. Note that only the case of a rotationally symmetric magnet was considered here. This tends to confirm the previous hypothesis made based on the section, Scaling Analysis for Radiated Angular Momentum. That scaling analysis found that there likely was angular momentum radiated to infinity for the original experiment, and we know on theoretical grounds that that experiment does produce a torque. In contrast, the variant experiment definitely did not have angular momentum radiated to infinity, and it did not produce a torque.

COMPUTATION OF RADIATED ANGULAR MOMENTUM

In this section, the angular momentum in the far field for the Feynman Disk experiment will be computed. The current source is assumed to be sinusoidal. As a result the angular momentum will also have a sinusoidal variation in time, with the same frequency. Before performing a careful computation, the physical reasons for expecting this term to be non zero will be reviewed. This will serve as an aid in interpreting the specific terms that will appear in the computation.

An accelerating charge radiates an electromagnetic field. The current going around the circular path has a centripetal acceleration and an acceleration due to the current changing in time. It can be shown that the centripetal acceleration is not the cause of the effect we are interested in. This follows from the observation that for a given current flow the radiated electric field due to the centripetal acceleration is not determined. In fact, it can be made to go to zero by increasing the number of charges moving and decreasing their speed. Thus, we are interested in the acceleration due to the current changing in time. This result is to be expected, since that is what produces the electric field at the outer ring and thus the torque.

In the non relativistic limit, the radiating part of the electric field produced (See Ref[6] Eqn.(14.14)) by an accelerating charge is

$$\vec{E} = \frac{e}{c^2 R} \hat{n} \times (\hat{n} \times \dot{\vec{v}}) |_{ret}$$

Consider the electric field produced in the plane of the current loop, by the current on the near side of that loop at a distance R. The electric field produced is in the opposite direction as the acceleration, which is tangent to the current loop. On the far side of the current loop, the acceleration is in the opposite direction as on the near side. If one ignores the retardation effect, then there is a cancellation that occurs and the electric field is reduced to a magnitude of R^{-2} , because for a loop of radius a,

$$\frac{1}{R} - \frac{1}{R+2a} = \frac{2a}{R(R+2a)} \approx 2aR^{-2} \quad \text{for } R \gg a$$

On the other hand, the radius of the loop might be one quarter of a wavelength at the frequency of oscillation; i.e. we might have,

$$a = \lambda/4 \quad \text{where } \lambda = 2\pi c/\omega$$

When the retardation is taken into account the electric fields produced by the opposite sides of the current loop add constructively. This gives a radiation field that decays as $1/r$ for large observation distances r. The magnetic field produced by this loop is perpendicular to both the electric field produced and its direction of propagation, so that the magnetic field must be in the $\pm \hat{z}$ direction. Remember that we have assumed the

observation is made in the plane of the current loop. The electric field resulting from the static charges on the outer loop is in the radial direction, so the (linear) momentum in the electric/magnetic field is in the $\hat{n} \times \hat{z} = \pm \hat{\phi}$ direction. Thus, this linear momentum produces angular momentum about the z-axis, as desired.

Although this suggests that we are on the right track, there is a problem. The radius of the outer (charge carrying) loop did not enter into this computation while that radius did enter into the computation of the torque. Thus, it appears that we will not be able to show this angular momentum is equal to the angular impulse (i.e., the time integral of the torque) that is applied to the disk. A more careful calculation of the radiated angular momentum is needed.

To calculate the angular momentum in the far field ($r' \gg a$), one notes that in that far field the leading terms in the vector potential are

$$\begin{aligned} \vec{A}(r, \theta, \phi, t) = & \frac{2a}{c} I_0 \phi \int_0^\pi d\phi' \frac{\cos\phi'}{r} \left(1 + \frac{a}{r} \cos\phi' \sin\theta\right) \\ & \left\{ \sin\left(\omega\left(t - \frac{r}{c}\right)\right) \cos\left(2\pi \frac{a}{\lambda} \cos\phi' \sin\theta\right) \right. \\ & \left. + \cos\left(\omega\left(t - \frac{r}{c}\right)\right) \sin\left(2\pi \frac{a}{\lambda} \cos\phi' \sin\theta\right) \right\} \end{aligned}$$

It can be seen that there are two terms here. The in-phase term varies according to the sine of ω times the retarded time, and the quadrature term varies as the cosine of ω times the retarded time. Note that the integral of even powers of the $\cos\phi'$ from zero to π is non zero while the integral of odd powers of $\cos\phi'$ is exactly zero. Using this result it can be seen that for small loops ($a/\lambda \Rightarrow 0$), the in-phase term has a leading distance behavior of r^{-2} (up to terms of order $[a/\lambda]^2$). It can also be seen that the quadrature term has a leading distance behavior of r^{-1} , with a size scaling as a/λ .

If we then take $a/\lambda = 1/4$, then the quadrature term gives a “large” result which scales as r^{-1} . This has the phase that would be expected by our earlier argument. It is out of phase with respect to the distance r to the center of the current loop, but is in-phase with respect to the actual distance from the current element to the observation point.

As a special case one might consider the vector potential produced on the plane where $\theta' = \pi/2$ by the current loop of radius $a = \lambda/4$. Two of the four terms above are then zero and two are nonzero. This can be proven by noting that $\cos((\pi/2)\cos\phi')$ is even about $\phi' = \pi/2$ and that $\sin((\pi/2)\cos\phi')$ is odd about $\phi' = \pi/2$. For that case, the vector potential simplifies to

$$A(r, \theta = \pi/2, \phi, t) = \frac{2a}{c} I_0 \phi \left\{ \frac{1.78073}{r} \cos\left(\omega\left(t - \frac{r}{c}\right)\right) + \frac{0.38a}{r^2} \sin\left(\omega\left(t - \frac{r}{c}\right)\right) \right\}$$

One may conclude from this that a spherical surface of radius r sees, at any one time, angular momentum being radiated through it. Thus, one may be tempted to conclude that this angular momentum is coming from the disk, and corresponds to the angular momentum being shed as an angular impulse is produced. This conclusion would be very satisfying, and would explain the observed angular impulse. It would be all the more satisfying since it would provide a way to study the Abraham/Minkowski paradox by observing radiated angular momentum for bodies with dielectric and magnetic materials. Unfortunately, the conclusion that the angular momentum passing through the surface at one time corresponds to an angular impulse would be incorrect.

The angular momentum passing through the spherical surface at any one time originates at different times, so it does not correspond to the angular impulse generated at one specific time. Consider one specific time and the angular momentum passing through the spherical surface at that time. For a current loop with a radius of a quarter wavelength ($a=\lambda/4$), consider the part of the angular momentum at that time resulting from the current on the near side of the loop and the part resulting from the current on the far side. One part came when an angular impulse was being created in one direction and the other part (from the other side of the loop) was created when an angular impulse was being created in the opposite direction. That is, the time difference is one-half a period, or a time different by $\lambda/2c$.

It is possible to look only at the in-phase part of the vector potential. In that case and for current loops of small radius a , the in-phase part of the vector potential decays with distance as $(1/r^2)$. Thus, this part of the vector potential decays so fast that it cannot cause angular momentum to be radiated to infinity. The proof of this simply repeats the arguments given in the section “Scaling Analysis for Radiated Angular Momentum,” while using the fact that now the radiated magnetic field from one particular time scales as $(1/r^2)$.

LOCAL ANGULAR MOMENTUM ANALYSIS

The results just found are subtle enough that it is worth verifying them by a different calculation. The local angular momentum for the original Feynman Disk experiment will be considered in this section. It should be noted that even seemingly simple calculations can become impossibly difficult when the exact form of electric and magnetic fields is taken into account. Up to this point, the difficulty in the computation of the angular momentum in the electric/magnetic field has been managed by only examining its far-field form. However, now that will change as we perform a computation of the local angular momentum at a fixed time. Let us begin by counting the integrations that must be performed. One integration will be over all time for the sources (the current and the charge). There are two more integrations over sources, one for the current and one for the charge. The result of those integrals is the angular momentum, which exists over a three dimensional space. Thus, in all, one must compute a six-dimensional integral. This is a complex problem to solve.

Previously, a sinusoidal current was used. Instead, here the current will have an initially constant value of I_0 in magnitude, and it will then decrease to zero and stay at zero for all later time. This form of the current will allow us to make a simplification. In computing the angular momentum, there is an integral over source time with a delta function giving the relation between retarded time and position. Since the current is initially constant, this integral can be performed trivially using the delta function. The computation of the angular impulse involves only evaluating the electric field on the outer ring, so there are fewer integrals than for the angular momentum. An additional simplification results from the fact that the torque and angular momentum vary linearly with the current. This results because the magnetic field and the electric field due to this current vary linearly with it, whereas the electric field due to static charges is independent of this current.

For the angular impulse calculation, there is a torque involving a derivative of the current flowing with respect to time. Integrating the torque produced over time allows this time derivative to be replaced by the total change in the current. That is, the total torque produced by the current decaying to zero depends only on the total change in the current, not on how the current changes in time. We find that, about the z-axis, the total angular impulse produced, AI , is

$$AI = z I_0 \frac{\alpha a Q}{c^2} \int_0^{2\pi} d\phi \frac{\cos\phi}{\sqrt{\alpha^2 + 1 - 2\alpha \cos\phi}}$$

Here “a” is the radius of the inner current carrying ring, “ αa ” is the radius of the outer ring that carries a total charge of Q , and I_0 is the initial current.

The total change in the angular momentum is exactly the initial angular momentum minus the final value (which is zero). This way of looking at the change results from removing the integral over time along with the delta function that enforces retarded time. This still involves a five-dimensional integral, two for the sources and three for the location of the angular momentum in three dimensional space. We will use the fact that both sources are rotationally symmetric to remove two of these integrals.

The rotational symmetry property is expressed in spherical coordinates by the, identity (Eqn. (3.70) in Ref[6]):

$$\frac{1}{|x - x'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{l,m}^*(\theta', \phi') Y_{l,m}(\theta, \phi)$$

The result for integrating a uniform charge around the ring follows from integrating the above formula (using formulas such as Eqn. (3.53) in Ref[6])

$$\int_0^{2\pi} d\phi \frac{1}{|x - x'|} = 2\pi \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l(\cos\theta') P_l(\cos\theta)$$

Notice that the source is in the plane where $\theta=\pi/2$ and thus where $\cos(\pi/2) = 0$. Applying the binomial expansion to Rodrigues's Formula (Eqn.(8.6.18) in Ref[7]) provides the numerical values of the Legendre functions $P_l(\theta)$ for $\theta=0$. Using that result, one finds that the above formula simplifies to

$$\int_0^{2\pi} d\phi \frac{1}{|x-x'|} = 2\pi \sum_{l=0}^{\infty} \frac{r_{<}^{2l}}{r_{>}^{2l+1}} P_{2l}(\cos\theta) \frac{(-1)^l (2l)!}{2^{2l} (l!)^2}$$

The voltage produced by the ring of static charge results from multiplying the result of this formula by the charge Q and dividing by the circumference ($2\pi\alpha a$). The electric field may then be found by taking the gradient of that voltage.

This series expansion for the electric field due to the charge ring separates the radial variable from the angular variables. This allows these integrals to be performed separately, resulting in great simplifications which will soon be evident. To that end, the magnetic field produced by the current ring will also be written using a similar expansion. The expansion for the magnetic field is given by Eqn.(5.48) and (5.49) in Reference [6] and is

$$B_r = \frac{2\pi I_0 a}{cr} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n n!} \frac{r_{<}^{2n+1}}{r_{>}^{2n+2}} P_{2n+1}(\cos\theta)$$

$$B_\theta = \frac{-\pi I_0 a^2}{c} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \frac{-(2n+2)}{(2n+1)a^3} \left(\frac{r}{a}\right)^{2n} P_{2n+1}^1(\cos\theta) \quad \text{for } r < a$$

$$B_\theta = \frac{-\pi I_0 a^2}{c} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \frac{1}{r^3} \left(\frac{a}{r}\right)^{2n} P_{2n+1}^1(\cos\theta) \quad \text{for } r > a$$

The angular momentum in the electric/magnetic field is in the \hat{z} direction and has a magnitude AM_I given by the sum of two parts

$$AM_I = AM_I^1 + AM_I^2$$

where these parts are given by

$$AM_I^1 = \frac{1}{2c} \int dr' d\theta' r' \sin\theta' r'^2 \sin\theta' (E_r B_\theta)$$

$$AM_I^2 = \frac{1}{2c} \int dr' d\theta' r' \sin\theta' r'^2 \sin\theta' (-E_\theta B_r)$$

The progress that we have made so far will be evident if one substitutes our formulas for the electric and magnetic fields into these formulas for the angular momentum. Rather than six integrals, there are now only two integrals. The integral over azimuthal angles, ϕ , has been performed. Two more of the integrals were replaced by a double sum over integer indices. Surprisingly, further simplifications are possible. We have managed to derive the formula

$$\int_0^\pi d\theta' \sin\theta' P_{2l}(\cos\theta') P_{2n+1}^1(\cos\theta') = \delta_{l,n} \frac{-4(2n+1)(n+1)}{(4n+1)(4n+3)} + \delta_{l,n+1} \frac{4(2n+1)(n+1)}{(4n+3)(4n+5)}$$

Using this formula in the result for AM_I not only removes the integral over θ' , it also reduces the double sum (i.e. over two indices) to a single sum (i.e. a sum over one index, albeit with two terms due to the two delta functions above). Thus, a formula results for AM_I which involves an integral over r' and a sum over an integer index. Since the integral over r' involves only powers of r' , that integral may easily be computed.

The result of all of these simplifications is a formula for the two parts of AM_I that involves only a summation over an integer index. It is nevertheless a somewhat complicated formula, since there are six complicated terms within that summation. The reason for this is that there are two delta functions above and three physical regions. These regions are those with a radius smaller than that of both rings, a radius in between that of the two rings, and a radius larger than that of both rings. For each region the expansions for the electric and magnetic fields differ. The calculation thus becomes quite involved and requires a large number of pages to carry out. The result for the first part of the angular momentum is

$$AM_I^1 = \frac{-2\pi}{4c} \frac{I_0 a Q}{c} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)!!}{2^n (n+1)!} \times \{$$

$$\left[\frac{(-1)^n (2n)!}{2^{2n} (n!)^2} \frac{-4(2n+1)(n+1)}{(4n+1)(4n+3)} \alpha^{-(2n+1)} \left(-\frac{(2n+2)2n}{(2n+1)(4n+3)} + 2n \log \alpha - \frac{2n+1}{4n+1} \right) \right] +$$

$$\left[\frac{(-1)^{n+1} (2n+2)!}{2^{2n+2} ((n+1)!)^2} \frac{4(2n+1)(n+1)}{(4n+3)(4n+5)} \alpha^{-(2n+3)} \left(-\frac{(2n+2)(2n+2)}{(2n+1)(4n+5)} + \frac{(2n+2)(\alpha^2 - 1)}{2} - \alpha^2 \frac{2n+3}{4n+3} \right) \right] \}$$

This series is an asymptotic series, not a convergent one. The reason is that there are singularities at each ring. That is, the series that had been used for the electric field and for the magnetic field are convergent except exactly on the ring that holds the corresponding sources. However, in deriving this series for AM_I , we performed integrals over r' with endpoints for the integration at these singularities.

At this point, the most satisfying conclusion to our work would be to combine the series solutions for AM_I^1 and AM_I^2 , to get a series solution for AM_I . Then, we could expand the elliptic integral appearing in the formula for the angular impulse AI and show

that term by term the series expansions for AI and AM_I are identical. We did not have time to do so. Instead, we verify the equality by a numerical test.

Treating the series for AM_I^1 as an asymptotic series, we observe that taking the sum for $n=0$ to 10 provides a reasonable answer. That is, the sum appears (numerically) to approach a limit near ten, and for much larger values begins to change again. As a test case, we take $\alpha=3$. That is, we assume that the charge carrying ring has a radius three times that of the current carrying ring. We also note that the various integrals may be performed numerically using the Mathematica software. The series will also be summed using Mathematica. We find that for $\alpha=3$:

$$AI = \frac{aQ}{c^2} \times 1.09413 \quad \text{by numerical integration}$$

$$AM_I^1 = \frac{aQ}{c^2} \times 0.68291 \quad \text{both by numerical integration and summing the series}$$

$$AM_I^2 = \frac{aQ}{c^2} \times 0.41087 \quad \text{by numerical integration}$$

$$AI = 1.09413 \approx 1.09378 = AM_I^1 + AM_I^2$$

The results for AM_I^1 by numerical integration and by summing the series agree to five digits. That is very satisfying. Also, the magnitude of the angular impulse and of the angular momentum agree to within the fourth digit. Given the nature of the asymptotic series, that is to be taken as a confirmation of the equality of the angular impulse and the angular momentum within our expected accuracy.

COMPARISON OF THE RESULTS ACHIEVED TO THE RESEARCH PLAN

The stated goal of this research had been to perform a computational experiment that compares the torques produced by the Abraham formulation, the Minkowski formulation, and a radiated field analysis. Our plan of research had explicitly made an assumption. That assumption was that the angular momentum in the radiated electromagnetic field equals minus the mechanical angular momentum produced. Based on that assumption, the plan was to examine configurations with and without dielectric/magnetic materials. The plan was to use the experience gained to identify configurations that would produce a relatively large torque (compared to the torque produced by other configurations). This plan has been followed as much as possible. However, some changes were necessary since the underlying assumption of this plan are shown to be incorrect.

In formulating this plan, we knew that loop antennas are often used as electromagnetic antennas. We knew that they produce electromagnetic fields that radiate, i.e., they decay with distance as $1/r$. It was found, as expected, that a spherical shell at a

large distance from the Feynman Disk experiment has angular momentum passing through it. It was found, also as expected, that at large distances the total amount of angular momentum per time passing through it is independent of the diameter of that spherical shell. Finally, it was found as expected that this amount of angular momentum per time varies sinusoidally, with the same frequency as the sinusoidal current on the inner ring.

The surprise in this research was that the radiated angular momentum is not associated with the angular impulse (torque times time) that was applied to the disk. The first hint that this is true came when we noticed that the radiated angular momentum (more accurately, its asymptotic form for large distances) did not depend on the radius of the ring holding the charges. However, the angular impulse did depend on this radius. In fact, as the radius of the charge-carrying ring is reduced sufficiently, the sign of the angular impulse changes while the radiated angular momentum is unchanged. Thus, the conservation law must not equate the angular impulse to only the radiated angular momentum. This led us to examine the non-radiating part of the angular momentum in the electric/magnetic field.

A calculation was made of the local part of the angular momentum in the electric/magnetic field using an initially steady current that then decayed to zero. Thus, an angular impulse was only produced in one direction. This provided a situation where the change in current could be related to the total angular momentum in space. This included the angular momentum that was both nearby and radiating. The calculation of the resulting angular momentum in the electric/magnetic field showed that it was all in the nearby region and that there was no radiating angular momentum. This illustrates a difficulty in dealing with the hidden momentum that exists internal to matter. In the case of magnetic materials, this may involve, for example, the current due to an electron circling an atom. If the actual motion of atoms due to dielectric or magnetic materials occurs with a delay, then so does this hidden momentum. Accounting for this can be difficult.

We found that when the current loop used is about a half a wavelength in diameter, its radiation is enhanced. This was to be expected, based on standard results in antenna theory. It was found that both the radiation and the radiated angular momentum were enhanced by using a loop of that size. In our research plan, we had expected that this could be used to design an experiment with an enhanced angular impulse. However, since radiated angular momentum is not directly related to the angular impulse, this proved not to be possible.

SUMMARY

We observed that for the Feynman Disk experiment, the angular momentum in the electromagnetic field, which is produced during a given time interval, is local and approaches zero strength as it enters the radiation (far-field) zone. This result is in contrast to that for a thin spherical shell of large radius centered on the disk. Such a shell contains a non-zero amount of angular momentum, which remains constant in magnitude

as the radius of the shell increases. We found that these contrasting results are consistent, since the angular momentum in the electromagnetic field for the region within such a shell is due to sources at different times.

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